A NOTE ON THE PAPER BY V.L.DOBROVOL'SKII "ON THE APPLICATION OF COMPLEX VARIABLES TO THE PLANE PLASTIC STRAIN"

(ZAMECHANIE K RABOTE V.L.DOBROVOL'SKOGO "O PRIMENENII KOMPLEKSNYKH PEREMENNYKH K PLOSKOI PLASTICHESKOI DEFORMATSII")

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B.D.ANNIN
(Novosibirsk)

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The necessary and sufficient conditions for a stress function $F(x,y)$ to be biharmonic in the plastic region D of the (x,y) plane were derived by Dobrovol'skii in [1]. These conditions are as follows. Let the function $\theta = \theta(x,y)\ (x,y) \in D$ be defined in terms of $F(x,y)$ from Equation

$$\tan\theta(x,y) = \frac{2\tau}{\sigma_y - \sigma_x} \qquad \left(\sigma_x = \frac{k}{2}\frac{\partial^2 F}{\partial y^2},\ \ \sigma_y = \frac{k}{2}\frac{\partial^2 F}{\partial x^2},\ \ \tau = -\frac{k}{2}\frac{\partial^2 F}{\partial x\,\partial y}\right)$$

Here σ_x, σ_y, τ are components of the stress tensor, k is the yield point in pure shear. In the region D the function $\theta(x,y)$ satisfies the system of equations

$$\frac{\partial^2\theta}{\partial x^2} - \frac{\partial^2\theta}{\partial y^2} - 2\frac{\partial\theta}{\partial x}\frac{\partial\theta}{\partial y} = 0, \qquad 2\frac{\partial^2\theta}{\partial x\,\partial y} + \left(\frac{\partial\theta}{\partial x}\right)^2 - \left(\frac{\partial\theta}{\partial y}\right)^2 = 0 \tag{1}$$

The following function is a particular solution of the above system

$$\theta^*(x,y) = -2\tan^{-1}\frac{y-y_0}{x-x_0} + \theta_0 \qquad (x_0, y_0, \theta_0 = \text{const})$$

The biharmonic property of the plastic stress function, to which $\theta^*(x,y)$ corresponds for $x_0 = y_0 = \theta_0 = 0$, has been substantially utilized by Galin in [2]. Let us prove that there are no solutions of the system (1) different from $\theta^*(x,y)$. Obviously, $\theta(x,y)$ is an analytic function with respect to x, y. Differentiating (1) with respect to x and y we determine all the third derivatives of the function $\theta(x,y)$. Then, from the condition

$$\frac{\partial}{\partial x}\frac{\partial^3\theta}{\partial y^3} = \frac{\partial}{\partial y}\frac{\partial^3\theta}{\partial x\,\partial y^2}$$

we find

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0 \tag{2}$$

Equation (2) also follows from the condition

$$\frac{\partial}{\partial x}\frac{\partial^3\theta}{\partial x^2\,\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial^2\theta}{\partial x^3}\right)$$

From Equations (1) we form Equation

$$\left[\left(\frac{\partial\theta}{\partial y}\right)^2 - \left(\frac{\partial\theta}{\partial x}\right)^2\right]\left[\frac{\partial^2\theta}{\partial x^2} - \frac{\partial^2\theta}{\partial y^2}\right] - 4\frac{\partial\theta}{\partial x}\frac{\partial\theta}{\partial y}\frac{\partial^2\theta}{\partial x\,\partial y} = 0 \qquad (3)$$

Applying the Legendre transformation to Equations (2) and (3), i.e. introducing the new variables ξ, η and a new function $\Phi(\xi,\eta)$ according to Formulas $\xi = \partial\theta / \partial x$, $\eta = \partial\theta / \partial y$, $\Phi = x\xi + y\eta - \theta$ and transferring to polar coordinates in the (ξ,η) plane, we find

$$\frac{\partial^2\Phi}{\partial r^2} = 0, \qquad \frac{\partial^2\Phi}{\partial\varphi^2} + r\frac{\partial\Phi}{\partial r} = 0$$

Hence follows

$$\theta = c \tan^{-1}\frac{y-b}{x-a} + d \qquad (a, b, c, d = \text{const}) \qquad (4)$$

Substituting (4) into the first Equation of (1) we find $c = -2$. Thus, there are no solutions of the system of equations (1) different from $\theta^*(x,y)$.

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